3D Shape Design to Maximize Volume

Drexel GK-12 Program
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Problem:

Abe Simpson seeks a treasure buried at the bottom of a lake. You need to get it before Mr. Burns.
Goal:

Imagine each piece of popcorn is a “weight”. Design a 3-D polygon that will sink to the bottom of the lake the fastest by maximizing volume.

Note: Keep in mind that density, not volume or mass determines if an object will sink.
Task Instructions:

- Form groups of 3-4
- Each person makes different 3-D shapes out of a single sheet of construction paper
- Test to see which holds the most popcorn
- Calculate the volume of each object in your group
Supplies:

• Construction paper
• Scissors
• Scotch Tape
• Ruler
• Popcorn
3D Object Examples

(Tip: Use regular polygons)
What did you find?

Which shape held the most popcorn?
• Next Steps...

Measure your shapes to find the surface area and volume of each object.

Formulas for Surface Area and Volume of 3D Figures

<table>
<thead>
<tr>
<th>Name</th>
<th>Surface Area</th>
<th>Volume</th>
</tr>
</thead>
<tbody>
<tr>
<td>Rectangular Prism</td>
<td>$SA = 2(lw + wh + lh)$</td>
<td>$V = lwh$</td>
</tr>
<tr>
<td>Cylinder</td>
<td>$SA = 2\pi r^2 + 2\pi rh$</td>
<td>$V = \pi r^2h$</td>
</tr>
<tr>
<td>Cone</td>
<td>$SA = \pi r^2 + \pi r l$</td>
<td>$V = \frac{\pi r^2h}{3}$</td>
</tr>
<tr>
<td>Sphere</td>
<td>$SA = 4\pi r^2$</td>
<td>$V = \frac{4\pi r^3}{3}$</td>
</tr>
<tr>
<td>Square Pyramid</td>
<td>$SA = b^2 + 2bl$</td>
<td>$V = \frac{b^2h}{3}$</td>
</tr>
</tbody>
</table>
Create a chart like the one below for a cylinder with a $SA = 375 \text{ cm}^2$ to explore optimal dimensions.

<table>
<thead>
<tr>
<th>Radius (cm)</th>
<th>Height (cm)</th>
<th>Volume (cm$^3$)</th>
<th>SA (cm$^2$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td></td>
<td></td>
<td>375</td>
</tr>
<tr>
<td>2</td>
<td></td>
<td></td>
<td>375</td>
</tr>
<tr>
<td>3</td>
<td></td>
<td></td>
<td>375</td>
</tr>
<tr>
<td>4</td>
<td></td>
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<td></td>
<td>375</td>
</tr>
<tr>
<td>8</td>
<td></td>
<td></td>
<td>375</td>
</tr>
</tbody>
</table>
Now you Try - Questions

1. What formulas did you need to use?

2. Which radius gives you the greatest volume?

3. Are there any relationships between radius, height, and maximal volume?

4. Do you think the dimensions that gave you the greatest volume are the optimal dimensions for a cylinder with a SA = 375 cm²?
Connections: Ballasting

Heavy material that is placed in the hold of a ship to enhance stability in water.
Connections - Seaperch: Underwater Robotics Competition

**Ballasting** is a critical component if one wishes to submerge and balance an underwater robot in order to complete various challenges.

Ballasting principles can help us sink quickly, too.
Now you Try - Extension

Volume alone won’t determine if an object will sink.

In future lessons, we will explore density of different objects compared to water.
Next Steps...

We need to know the volume and mass in order to eventually calculate density. But this lesson focuses on volume, a necessary component.

\[
\text{Density} = \frac{\text{Mass}}{\text{Volume}}
\]
Vocabulary:

Surface Area - sum of all the areas that cover the surface of an object
Volume - the quantity that a shape / object occupies or contains
Radius - distance from the edge of a circle to the center (1/2 of diameter)
Regular Polygon- shape with equal sides and angles
Irregular Polygon- shape that does not have equal sides and angles
Mass - often measured in weight (differences are beyond this lesson)
Dimensions- Number of measurements needed to represent an object, commonly length, width, and height
Substitution- putting numbers where letters are in an equation
Density- relative “heaviness” of objects with the same volume
(Density = Mass / Volume)
Supplementary Material: Advanced Extensions w/ Calculus
Example (rectangular prism with NO TOP):

S.A. = 48 = (area of base) + 4 (area of one side) = \( x^2 + 4 \cdot (xy) \)

\[ 4xy = 48 - x^2 \]

\[ y = \frac{(48-x^2)}{4x} \]

\[ y = \frac{48}{4x} - \frac{x^2}{4x} \]

\[ y = \frac{12}{x} - \frac{(1/4)x}{ } \]
Maximize Volume using Substitution

\[ V = (x) (x) (y) = x^2 y \]
\[ V = x^2 \left(12/x - (1/4)x\right) \]
\[ V = 12x - (1/4)x^3 \]

Now **Differentiate** this equation:

\[ V' = 12 - (1/4)3x^2 \]
\[ V' = 12 - (3/4)x^2 \]
\[ V' = (3/4) (16 - x^2) \]
\[ V' = (3/4) (4 - x) (4 + x) \]
\[ V' = 0 \]
\[ x = 4, -4 \text{ (can’t be negative)} \]

Now we can go back and solve for \( y \) and find the maximum volume....what did you get?
You may have noticed that for a cylinder, the volume is maximized with the $h = 2r$ (diameter)

We can use the formula for SA to determine optimal dimensions...

$$SA = 2\pi r^2 + 2\pi rh$$

Solve for $h$...
You have enough money to buy 600 cm\(^2\) of PVC pipe (with top and bottom). You want to fill them with lead shots, so they will sink quickly.

What dimensions would you use (radius and height) to maximize volume?
Now you Try - Extension

Volume alone won’t determine if an object will sink.

In future lessons, we will explore density of different objects compared to water.